

# RESIDUE-TYPE INDICES AND HOLOMORPHIC FOLIATIONS

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**ABSTRACT.** We investigate residue-type indices for germs of holomorphic foliations in the plane and characterize second type foliations — those not containing tangent saddle-nodes in the reduction of singularities — by an expression involving the Baum-Bott, variation and polar excess indices. These local results are applied in the study of logarithmic foliations on compact complex surfaces.

## 1. INTRODUCTION

In 1997 M. Brunella [3] proved the following result:

**Theorem.** Let  $\mathcal{F}$  be a non-dicritical germ of holomorphic foliation at  $(\mathbb{C}^2, p)$  and let  $S$  denote the union of all its separatrices. If  $\mathcal{F}$  is a generalized curve foliation then

$$\text{BB}_p(\mathcal{F}) = \text{CS}_p(\mathcal{F}, S) \quad \text{and} \quad \text{GSV}_p(\mathcal{F}, S) = 0.$$

The foliation  $\mathcal{F}$  is said to be a *generalized curve* if there are no saddle-nodes in its reduction of singularities. This concept was introduced in [4] and delimits a family of foliations whose topology is closely related to that of their separatrices — local invariant curves — which in this case are all analytic. In the statement of the theorem, *non-dicritical* means that the separatrices are finite in number. Further, BB, CS and GSV stand for, respectively, the Baum-Bott, the Camacho-Sad and the Gomez-Mont-Seade-Verjovsky indices.

Generalized curve foliations are part of the broader family of *second type foliations*, introduced by J.-F. Mattei and E. Salem in [17]. Foliation in this family may admit saddle-nodes in the reduction of singularities, provided they are not *tangent saddle-nodes* (Definition 2.1). A second type foliation satisfies the remarkable property of getting reduced once its set of separatrices — including the formal ones — is desingularized. Recently, second type foliations have been the object of some works. We should mention [10] — which deals with the “realization problem”, that is, the existence of foliations with prescribed reduction of singularities and projective holonomy representations, [11] — which studies local polar invariants and applications to the study of the Poincaré problem for foliations — and [19] — where equisingularity properties are considered. Our main goal in this article is to give a characterization of second type foliations by means of residue-type indices, providing a generalization of Brunella’s result.

Our work is strongly based on the notion of *balanced set* or *balanced equation* of separatrices ([10] and Definition 2.3). This is a geometric object formed by a finite set of

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separatrices with weights — possibly negative, corresponding to poles — that, in the non-dicritical case, coincides with the whole separatrix set. A balanced set of separatrices provides a control of the algebraic multiplicity of the foliation and, for second type foliations, it actually determines it (Proposition 2.4). In the text, we will preferably see this object as a divisor of formal curves  $\mathcal{B}$  — a *balanced divisor* of separatrices — having a decomposition  $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$  as a difference of effective divisors of zeros and poles.

To a germ of foliation  $\mathcal{F}$  and a finite set of separatrices  $C$  — which can contain purely formal ones — we associate a triplet of residue-type indices: the afore mentioned CS-index and GSV-index, along with the *variation* index  $\text{Var}$  — that turns out to be the sum of the two first indices (definitions in [5], [12] and [13]; see equation (14) below). We then form a quadruplet of indices by including the *polar excess* index  $\Delta$  ([11] and Definition 3.1). This one is calculated by means of polar invariants and can be seen as a measure of the existence of saddle-nodes in the reduction process of  $\mathcal{F}$  (Theorem 3.2 and Proposition 3.5). All these indices are subject of a more detailed discussion in Section 3.

Let  $I_p(\mathcal{F}, C)$  denote some index in the quadruplet. In the non-dicritical case, if  $C$  is the curve formed by the complete set of separatrices, the index is said to be *total* and is denoted as  $I_p(\mathcal{F})$ . We extend the notion of total index to dicritical foliations, employing a balanced divisor of separatrices  $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$  in place of the curve  $C$  in the following way (Definition 3.4):

$$I_p(\mathcal{F}) := I_p(\mathcal{F}, \mathcal{B}_0) - I_p(\mathcal{F}, \mathcal{B}_\infty).$$

This definition is particularly well suited to the  $\text{Var}$ -index and to the  $\Delta$ -index, since both of them are additive in the separatrix set.

The main result of this article is the following:

**Theorem I.** *Let  $\mathcal{F}$  be a germ of holomorphic foliation at  $(\mathbb{C}^2, p)$ . Then  $\mathcal{F}$  is of second type if and only if*

$$\text{BB}_p(\mathcal{F}) = \text{Var}_p(\mathcal{F}) + \Delta_p(\mathcal{F}),$$

where  $\text{BB}_p(\mathcal{F})$  is the Baum-Bott index,  $\text{Var}_p(\mathcal{F})$  and  $\Delta_p(\mathcal{F})$  are the total variation and polar excess indices. Moreover,  $\mathcal{F}$  is a generalized curve foliation if and only if

$$\text{BB}_p(\mathcal{F}) = \text{Var}_p(\mathcal{F}).$$

Indeed, for an arbitrary foliation, we can evaluate the difference of the left and right sides of the expression in the theorem as a non-negative integer that assembles the contribution of tangent saddle-nodes along the reduction of singularities. This is done in Theorem 5.2, from which Theorem I is a corollary. In the non-dicritical case,  $\Delta_p(\mathcal{F}) = \text{GSV}_p(\mathcal{F})$  (Theorem 3.3) and  $\mathcal{F}$  is a generalized curve foliation if and only if  $\text{GSV}_p(\mathcal{F}) = 0$ . Theorem I thus recovers the statement of Brunella's theorem simultaneously providing its converse: a non-dicritical  $\mathcal{F}$  is a generalized curve foliation if and only if  $\text{BB}_p(\mathcal{F}) = \text{CS}_p(\mathcal{F})$ .

The article is structured as follows. In section 2 we present some basic definitions and properties of local foliations with a specific view on second type foliations. Section 3 is a brief review on residue-type indices, where we explain the case of formal separatrices and define the total index. In section 4 we introduce a new invariant, the *second variation* index — the sum of the variation and polar excess indices — and calculate its change by blow-up maps. Then, in section 5, we compare second variation and Baum-Bott indices (Theorem 5.2) and derive the proof of Theorem I. Next, as an application of Theorem I, we obtain in section 6 a characterization of non-dicritical logarithmic foliations in terms

of second type foliation, both in the complex projective plane (Proposition 6.1) and in the more general setting of projective surfaces with infinite cyclic Picard group (Proposition 6.2). We close this article by presenting, in section 7, numerical data of a pair of examples.

## 2. BASIC DEFINITIONS AND NOTATION

In order to fix a terminology and a notation, we recall some basic concepts of local foliation theory. Let  $\mathcal{F}$  be a holomorphic foliation with isolated singularities on a complex surface  $X$ . Let  $p \in X$  be a singular point of  $\mathcal{F}$ . In local coordinates  $(x, y)$  centered at  $p$ , the foliation is given by an analytic vector field

$$(1) \quad v = F(x, y) \frac{\partial}{\partial x} + G(x, y) \frac{\partial}{\partial y}$$

or by its dual 1-form

$$(2) \quad \omega = G(x, y)dx - F(x, y)dy,$$

where  $F, G \in \mathbb{C}\{x, y\}$  are relatively prime.

A *separatrix* for  $\mathcal{F}$  is an invariant formal irreducible curve, that is, an object given by an irreducible formal series  $f \in \mathbb{C}[[x, y]]$  satisfying

$$\omega \wedge df = (fh)dx \wedge dy$$

for some formal series  $h \in \mathbb{C}[[x, y]]$ . The separatrix is said to be *analytic* or *convergent* if we can take  $f \in \mathbb{C}\{x, y\}$ . It is said to be *purely formal* otherwise. We denote by  $\text{Sep}_p(\mathcal{F})$  the set of all separatrices of  $\mathcal{F}$  at  $p$ .

We say that  $p \in \mathbb{C}^2$  is a *reduced* or *simple* singularity for  $\mathcal{F}$  if the linear part  $Dv(p)$  of the vector field  $v$  in (1) is non-zero and has eigenvalues  $\lambda_1, \lambda_2 \in \mathbb{C}$  fitting in one of the two cases:

- (i)  $\lambda_1 \lambda_2 \neq 0$  and  $\lambda_1 / \lambda_2 \notin \mathbb{Q}^+$  (*non-degenerate* or *complex hyperbolic* singularity).
- (ii)  $\lambda_1 \neq 0$  and  $\lambda_2 = 0$  (*saddle-node* singularity).

In case (i), there are analytic coordinates in  $(x, y)$  in which  $\mathcal{F}$  is induced by the equation

$$(3) \quad \omega = x(\lambda_1 + a(x, y))dy - y(\lambda_2 + b(x, y))dx,$$

where  $a, b \in \mathbb{C}\{x, y\}$  are non-units, so that  $\text{Sep}_p(\mathcal{F})$  is formed by two transversal analytic branches given by  $\{x = 0\}$  and  $\{y = 0\}$ . In case (ii), up to a formal change of coordinates, the saddle-node singularity is given by a 1-form of the type

$$(4) \quad \omega = y(1 + \lambda x^k)dx + x^{k+1}dy,$$

where  $\lambda \in \mathbb{C}$  and  $k \in \mathbb{Z}_{>0}$  are formal invariants [16]. The curve  $\{x = 0\}$  is an analytic separatrix, called *strong*, whereas  $\{y = 0\}$  corresponds to a possibly formal separatrix, called *weak* or *central*. The integer  $k+1 > 1$  is called the *tangency index* of  $\mathcal{F}$  with respect to the weak separatrix, *weak index* for short, and will be denoted as  $\text{Ind}_p^w(\mathcal{F})$ .

Let  $\pi : (\tilde{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, p)$  be a composition of blow-up maps. The divisor  $\mathcal{D}$  is a finite union of components which are embedded projective lines, crossing normally at *corners*. If  $\mathcal{F}$  is the foliation defined by the 1-form  $\omega$ , we denote by  $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$  the *strict transform* of  $\mathcal{F}$ , the germ of foliation on  $(\tilde{X}, \mathcal{D})$  defined locally by  $\pi^* \omega$ , obtained after cancelling the one-dimensional singular components. For a uniform analysis, we include the possibility

of  $\pi$  being the identity map and, abusing notation, we set in this case  $\tilde{X} = \mathbb{C}^2$ ,  $\mathcal{D} = \{p\}$  and  $\tilde{\mathcal{F}} = \mathcal{F}$ .

With respect to the the divisor  $\mathcal{D}$ , the foliation  $\tilde{\mathcal{F}}$  at a point  $q \in \mathcal{D}$  can be:

- *regular*, if there are local analytic coordinates  $(x, y)$  at  $q$  such that  $\mathcal{D} \subset \{xy = 0\}$  and  $\tilde{\mathcal{F}} : dx = 0$ ;
- *singular*, if it is not regular;
- *reduced* or *simple*, if  $q$  is a reduced singularity for  $\tilde{\mathcal{F}}$  and  $\mathcal{D} \subset \text{Sep}_q(\tilde{\mathcal{F}})$ .

For simplicity, we employ the terminology  $\mathcal{D}$ -regular,  $\mathcal{D}$ -singular and  $\mathcal{D}$ -reduced. When  $\mathcal{D} = \{p\}$ , these notions coincide with the ordinary concepts of regular point, singular point and reduced singularity. We say that  $\pi : (\tilde{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, p)$  is a *reduction of singularities* or *desingularization* for  $\mathcal{F}$  if all points  $q \in \mathcal{D}$  are either  $\mathcal{D}$ -regular or  $\mathcal{D}$ -reduced singularities. There always exists a reduction of singularities [21, 4]. Besides, there exists a *minimal* one, in the sense that it factorizes any other reduction of singularities by an additional sequence of blow-ups. All along this text, reductions of singularities are supposed to be minimal.

Given a germ of foliation  $\mathcal{F}$  at  $(\mathbb{C}^2, p)$  we introduce the set  $\mathcal{I}_p(\mathcal{F})$  of *infinitely near points* of  $\mathcal{F}$  at  $p$ . This is defined in a recursive way along the reduction of singularities of  $\mathcal{F}$ . We do as follows. Given a sequence of blow-ups  $\pi : (\tilde{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$  — an intermediate step in the reduction process — and a point  $q \in \mathcal{D}$  we set:

- if  $\tilde{\mathcal{F}}$  is  $\mathcal{D}$ -reduced at  $q$ , then  $\mathcal{I}_q(\tilde{\mathcal{F}}) = \{q\}$ ;
- if  $\tilde{\mathcal{F}}$  is  $\mathcal{D}$ -singular but not  $\mathcal{D}$ -reduced at  $q$ , we perform a blow-up  $\sigma : (\hat{X}, \hat{\mathcal{D}}) \rightarrow (\tilde{X}, \mathcal{D})$  at  $q$ , where  $\hat{\mathcal{D}} = \sigma^{-1}(\mathcal{D}) = (\sigma^*\mathcal{D}) \cup D$  and  $D = \sigma^{-1}(q)$ . If  $q_1, \dots, q_\ell$  are all  $\hat{\mathcal{D}}$ -singular points of  $\hat{\mathcal{F}} = \sigma^*\tilde{\mathcal{F}}$  on  $D$ , then

$$\mathcal{I}_q(\tilde{\mathcal{F}}) = \{q\} \cup \mathcal{I}_{q_1}(\hat{\mathcal{F}}) \cup \dots \cup \mathcal{I}_{q_\ell}(\hat{\mathcal{F}}).$$

In order to simplify notation, we settle that a numerical invariant for a foliation  $\mathcal{F}$  at  $q \in \mathcal{I}_p(\mathcal{F})$  actually means the same invariant computed for the transform of  $\mathcal{F}$  at  $q$ . Context will make this clear.

For a fixed a reduction process  $\pi : (\tilde{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, p)$  for  $\mathcal{F}$ , a component  $D \subset \mathcal{D}$  can be:

- *non-dicritical*, if  $D$  is  $\tilde{\mathcal{F}}$ -invariant. In this case,  $D$  contains a finite number of simple singularities. Each non-corner singularity carries a separatrix transversal to  $D$ , whose projection by  $\pi$  is a curve in  $\text{Sep}_p(\mathcal{F})$ .
- *dicritical*, if  $D$  is not  $\tilde{\mathcal{F}}$ -invariant. The definition of reduction of singularities gives that  $D$  may intersect only non-dicritical components and that  $\tilde{\mathcal{F}}$  is everywhere transverse do  $D$ . The  $\pi$ -image of a local leaf of  $\tilde{\mathcal{F}}$  at each non-corner point of  $D$  belongs to  $\text{Sep}_p(\mathcal{F})$ .

Denote by  $\text{Sep}_p(D) \subset \text{Sep}_p(\mathcal{F})$  the set of separatrices whose transforms by  $\pi$  intersect the component  $D \subset \mathcal{D}$ . If  $B \in \text{Sep}_p(D)$  with  $D$  non-dicritical,  $B$  is said to be *isolated*. Otherwise, it is said to be a *dicritical separatrix*. This engenders the decomposition  $\text{Sep}_p(\mathcal{F}) = \text{Iso}_p(\mathcal{F}) \cup \text{Dic}_p(\mathcal{F})$ , where notations are self evident. The set  $\text{Iso}_p(\mathcal{F})$  is finite and contains all purely formal separatrices. It subdivides further in two classes: *weak* separatrices — those arising from the weak separatrices of saddle-nodes — and *strong* separatrices — corresponding to strong separatrices of saddle-nodes and separatrices of non-degenerate singularities. On the other hand, if non-empty,  $\text{Dic}_p(\mathcal{F})$  is an infinite set of

analytic separatrices. A foliation  $\mathcal{F}$  is said to be *dicritical* when  $\text{Sep}_p(\mathcal{F})$  is infinite, which is equivalent to saying that  $\text{Dic}_p(\mathcal{F})$  is non-empty. Otherwise,  $\mathcal{F}$  is called *non-dicritical*.

Along the text, we would rather adopt the language of *divisors* of formal curves. More specifically, a *divisor of separatrices* for a foliation  $\mathcal{F}$  at  $(\mathbb{C}^2, p)$  is a formal sum

$$\mathcal{B} = \sum_{B \in \text{Sep}_p(\mathcal{F})} a_B \cdot B$$

where the coefficients  $a_B \in \mathbb{Z}$  are zero except for finitely many  $B \in \text{Sep}_p(\mathcal{F})$ . We denote by  $\text{Div}_p(\mathcal{F})$  the set of all these divisors, which turns into a group with the canonical additive structure. We follow the usual terminology and notation:

- $\mathcal{B} \geq 0$  denotes an *effective* divisor, one whose coefficients are all non-negative;
- there is a unique decomposition  $\mathcal{B} = \mathcal{B}_0 - \mathcal{B}_\infty$ , where  $\mathcal{B}_0, \mathcal{B}_\infty \geq 0$  are respectively the *zero* and *pole* divisors of  $\mathcal{B}$ ;
- the *algebraic multiplicity* of  $\mathcal{B}$  is  $\nu_p(\mathcal{B}) = \sum_{B \in \text{Sep}_p(\mathcal{F})} a_B$ .

Given a formal meromorphic equation  $\hat{F}$ , whose irreducible components define separatrices  $B_i$  with multiplicities  $\nu_i$ , we associate the divisor  $(\hat{F}) = \sum_i \nu_i \cdot B_i$ . A curve of separatrices  $C$ , associated to a reduced equation  $\hat{F}$ , is identified to the divisor  $C = (\hat{F})$ . Such an effective divisor is named *reduced*, that is, all coefficients are either 0 or 1. In general,  $\mathcal{B} \in \text{Div}_p(\mathcal{F})$  is reduced if both  $\mathcal{B}_0$  and  $\mathcal{B}_\infty$  are reduced effective divisors. A divisor  $\mathcal{B}$  is said to be *adapted* to a curve of separatrices  $C$  if  $\mathcal{B}_0 - C \geq 0$ . Finally, the usual intersection number for formal curves at  $(\mathbb{C}^2, p)$ , denoted by  $(\cdot, \cdot)_p$ , is canonically extended in a bilinear way to divisors of curves.

Let  $\mathcal{F}$  be a germ of foliation at  $(\mathbb{C}^2, p)$  with reduction process  $\pi : (\tilde{X}, \mathcal{D}) \rightarrow (\mathbb{C}^2, p)$  and let  $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$  be the strict transform foliation. A saddle-node singularity  $q \in \text{Sing}(\tilde{\mathcal{F}})$  is said to be a *tangent saddle-node* if its weak separatrix is contained in the exceptional divisor  $\mathcal{D}$ . We have the following definition [17]:

**Definition 2.1.** A foliation is in the *second class* or is of *second type* if there are no tangent saddle-nodes in its reduction process.

Given a component  $D \subset \mathcal{D}$ , we denote by  $\rho(D)$  its multiplicity, which coincides with the algebraic multiplicity of a curve  $\gamma$  at  $(\mathbb{C}^2, p)$  whose transform  $\pi^* \gamma$  meets  $D$  transversally outside a corner of  $\mathcal{D}$ . The following invariant is a measure of the existence of tangent saddle-nodes in the reduction of singularities of a foliation:

**Definition 2.2.** The *tangency excess* of  $\mathcal{F}$  is the number

$$\tau_p(\mathcal{F}) = \sum_{q \in \text{SN}(\mathcal{F})} \rho(D_q)(\text{Ind}_q^w(\tilde{\mathcal{F}}) - 1),$$

where  $\text{SN}(\mathcal{F})$  stands for the set of tangent saddle-nodes on  $\mathcal{D}$  and, if  $q \in \text{SN}(\mathcal{F})$ , we denote by  $D_q$  the component of  $\mathcal{D}$  containing its weak separatrix and by  $\text{Ind}_q^w(\tilde{\mathcal{F}}) > 1$  its weak index.

Of course,  $\tau_p(\mathcal{F}) \geq 0$  and, by definition,  $\tau_p(\mathcal{F}) = 0$  if and only if  $\text{SN}(\mathcal{F}) = \emptyset$ , that is, if and only if  $\mathcal{F}$  is of second type. We introduce the following object [10, 11]:

**Definition 2.3.** A *balanced divisor of separatrices* for  $\mathcal{F}$  is a divisor of the form

$$\mathcal{B} = \sum_{B \in \text{Iso}_p(\mathcal{F})} B + \sum_{B \in \text{Dic}_p(\mathcal{F})} a_B \cdot B,$$

where the coefficients  $a_B \in \mathbb{Z}$  are non-zero except for finitely many  $B \in \text{Dic}_p(\mathcal{F})$ , and, for each dicritical component  $D \subset \mathcal{D}$ , the following equality is respected:

$$\sum_{B \in \text{Sep}_p(D)} a_B = 2 - \text{Val}(D).$$

The integer  $\text{Val}(D)$  stands for the *valence* of a component  $D \subset \mathcal{D}$  in the reduction process, that is, it is the number of components of  $\mathcal{D}$  intersecting  $D$  other from  $D$  itself.

A balanced divisor  $\mathcal{B}$  is called *primitive* if, for every dicritical component  $D \in \mathcal{D}$  and every  $B \in \text{Sep}_p(D)$ , either  $-1 \leq a_B \leq 0$  or  $0 \leq a_B \leq 1$ . Recall that a balanced divisor  $\mathcal{B}$  is *adapted* to a curve of separatrices  $C$  if  $\mathcal{B}_0 - C \geq 0$ . A *balanced equation of separatrices* is a formal meromorphic function  $\hat{F}$  whose associated divisor is a balanced divisor. A balanced equation is *reduced*, *primitive* or *adapted* to a curve  $C$  if the same is true for the underlying divisor.

The tangency excess measures the extent that a balanced divisor of separatrices computes the algebraic multiplicity, as expressed in the following result [10]:

**Proposition 2.4.** *Let  $\mathcal{F}$  be a germ of singular foliation at  $(\mathbb{C}^2, p)$  with  $\mathcal{B}$  as a balanced divisor of separatrices. Denote by  $\nu_p(\mathcal{F})$  and  $\nu_p(\mathcal{B})$  their algebraic multiplicities. Then*

$$\nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1 + \tau_p(\mathcal{F}).$$

Moreover,

$$\nu_p(\mathcal{F}) = \nu_p(\mathcal{B}) - 1$$

if and only if  $\mathcal{F}$  is a second type foliation.

### 3. INDICES OF FOLIATIONS

In this section we briefly recall definitions and main properties of some indices associated to singular plane foliations, following the presentation in [3]. Some of these indices are calculated with respect to invariant analytic curves and we explain how to extend their definitions to formal invariant curves. We shall also present the *polar excess* index, introduced in [11]. In our exposition, invariant curves are identified with reduced divisors of separatrices. Calculations and definitions apply to germs of foliations lying on a complex surface, but we can transfer them to the complex plane by taking local analytic coordinates.

**3.1. The Baum-Bott index.** Let  $\mathcal{F}$  be a germ of foliation defined either by a holomorphic vector field  $v$  as in (1) or by a holomorphic 1-form  $\omega$  as in (2). If  $J(x, y)$  denotes the Jacobian matrix of  $(F, G)$  in the variables  $(x, y)$ , then the following residue defines the *Baum-Bott index* at  $p \in \text{Sing}(\mathcal{F})$  [1]:

$$\text{BB}_p(\mathcal{F}) = \text{Res}_p \left\{ \frac{(\text{tr} J)^2}{F \cdot G} dx \wedge dy \right\}.$$

For a reduced singularity with local models (3) and (4), this becomes:

$$(5) \quad \text{BB}_p(\mathcal{F}) = \begin{cases} \frac{(\text{tr} J(p))^2}{\det J(p)} = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + 2 & \text{if } p \text{ is non-degenerate;} \\ 2k + 2 + \lambda & \text{if } p \text{ is a saddle-node.} \end{cases}$$

On a compact surface  $M$ , the sum of Baum-Bott indices of a foliation  $\mathcal{F}$  is expressed in terms of the first Chern class of the normal bundle  $N_{\mathcal{F}}$  of the foliation [1, 2]:

$$(6) \quad \sum_{p \in \text{Sing}(\mathcal{F})} \text{BB}_p(\mathcal{F}) = c_1^2(N_{\mathcal{F}}).$$

**3.2. The Camacho-Sad index.** Let  $C$  be an invariant analytic curve for  $\mathcal{F}$  defined by a reduced function  $f \in \mathbb{C}\{x, y\}$ . Then there are germs  $g, k \in \mathbb{C}\{x, y\}$ , with  $k$  and  $f$  relatively prime, and a germ of analytic 1-form  $\eta$  such that

$$(7) \quad g\omega = kdf + f\eta$$

(see, for instance, [15, 22]). The Camacho-Sad index [5] is the residue

$$(8) \quad \text{CS}_p(\mathcal{F}, C) = -\frac{1}{2\pi i} \int_{\partial C} \frac{1}{k} \eta.$$

The integral is over  $\partial C = C \cap S^3$ , the link of  $C$  oriented as the boundary of  $C \cap B^4$ , where  $B^4$  is a small ball centered at  $0 \in \mathbb{C}^2$  and  $S^3 = \partial B^4$ . If  $C_1$  and  $C_2$  are  $\mathcal{F}$ -invariant curves without common components, then the following adjunction formula holds:

$$(9) \quad \text{CS}_p(\mathcal{F}, C_1 + C_2) = \text{CS}_p(\mathcal{F}, C_1, p) + \text{CS}_p(\mathcal{F}, C_2, p) + 2(C_1 \cdot C_2)_p.$$

A decomposition (7) also exists for a branch of formal separatrix  $B$  with formal equation  $f \in \mathbb{C}[[x, y]]$ , yielding  $g, k$  and  $\eta$  as formal objects. In this context, we can extend the definition of the Camacho-Sad index to  $B$  by taking  $\gamma(T)$ , a Puiseux parametrization for  $B$  such that  $\gamma(0) = p$ , and setting

$$\text{CS}_p(\mathcal{F}, B) = \text{Res}_{t=0} \gamma^* \left( \frac{1}{k} \eta \right).$$

Clearly, when  $B$  is convergent, this coincides with (8). Finally, the CS-index may be defined for a reducible curve of separatrices containing some purely formal branches by applying the adjunction formula (9).

The following result is known as the Camacho-Sad index Theorem [5]: if  $C \subset M$  is a compact curve invariant by a foliation  $\mathcal{F}$  on a complex surface  $M$ , then

$$(10) \quad \sum_{p \in \text{Sing}(\mathcal{F}) \cap C} \text{CS}_p(\mathcal{F}, C) = C \cdot C.$$

**3.3. The Gomez-Mont-Seade-Verjovsky index.** The decomposition (7) is also used to calculate the GSV-index (due to Gomez-Mont, Seade and Verjovsky, [12]) with respect to an  $\mathcal{F}$ -invariant curve  $C$ :

$$\text{GSV}_p(\mathcal{F}, C) = \frac{1}{2\pi i} \int_{\partial C} \frac{g}{k} d \left( \frac{k}{g} \right).$$

The adjunction formula now reads:

$$(11) \quad \text{GSV}_p(\mathcal{F}, C_1 + C_2) = \text{GSV}_p(\mathcal{F}, C_1) + \text{GSV}_p(\mathcal{F}, C_2) - 2(C_1 \cdot C_2)_p,$$



where  $C_1$  and  $C_2$  are  $\mathcal{F}$ -invariant curves without common components.

The extension of this definition to a purely formal branch of separatrix  $B$  is done as previously: take  $\gamma(T)$  a Puiseux parametrization for  $B$  such that  $\gamma(0) = p$  and set

$$\text{GSV}_p(\mathcal{F}, B) = \text{Res}_{t=0} \gamma^* \left( \frac{g}{k} d \left( \frac{k}{g} \right) \right) = \text{ord}_{t=0} \left( \frac{k}{g} \circ \gamma(t) \right).$$

Then, use the adjunction formula (11) in order to define the GSV-index for an invariant curve  $C$  containing some purely formal branches.

For the GSV-index, we can also state a result of global nature [2]: if the compact curve  $C \subset M$  is invariant by a foliation  $\mathcal{F}$  on a complex surface  $M$ , then

$$(12) \quad \sum_{p \in \text{Sing}(\mathcal{F}) \cap C} \text{GSV}_p(\mathcal{F}, C) = c_1(N_{\mathcal{F}}) \cdot C - C \cdot C.$$

**3.4. The variation index.** Each point  $q$  in a small punctured neighborhood of  $p \in \mathbb{C}^2$  is regular for  $\mathcal{F}$ . Then there exists a germ of holomorphic 1-form  $\zeta$  at  $q$  such that  $d\omega = \zeta \wedge \omega$ . If  $\zeta'$  is another such 1-form, we have that  $\zeta$  and  $\zeta'$  coincide over every leaf of  $\mathcal{F}$ . Therefore, in this punctured neighborhood, we can define a multi-valued 1-form, still denoted by  $\zeta$ , with single-valued restriction to each leaf of  $\mathcal{F}$ , satisfying the equation

$$d\omega = \zeta \wedge \omega.$$

The *variation index* [13] for an  $\mathcal{F}$ -invariant analytic curve  $C$  is defined as

$$\text{Var}_p(\mathcal{F}, C) = \frac{1}{2\pi i} \int_{\partial C} \zeta.$$

This index is additive in the separatrices of  $\mathcal{F}$ :

$$(13) \quad \text{Var}_p(\mathcal{F}, C_1 + C_2) = \text{Var}_p(\mathcal{F}, C_1) + \text{Var}_p(\mathcal{F}, C_2)$$

whenever  $C_1$  and  $C_2$  are  $\mathcal{F}$ -invariant curves without common components. Thus, for a divisor of separatrices  $\mathcal{B} = \sum_B a_B \cdot B$  we can define

$$\text{Var}_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \text{Var}_p(\mathcal{F}, B).$$

For an analytic invariant curve  $C$ , we have the relation

$$(14) \quad \text{Var}_p(\mathcal{F}, C) = \text{CS}_p(\mathcal{F}, C) + \text{GSV}_p(\mathcal{F}, C).$$

Now, when it comes to defining  $\text{Var}_p(\mathcal{F}, B)$  for a formal branch of separatrix  $B$ , the strategy followed for the CS and the GSV indices is unsuitable, since the 1-form  $\zeta$  does not define a formal object at  $p \in \mathbb{C}^2$ . However, knowing  $\text{CS}_p(\mathcal{F}, B)$  and  $\text{GSV}_p(\mathcal{F}, B)$  for a formal separatrix  $B$ , we can adopt formula (14) as a definition for  $\text{Var}_p(\mathcal{F}, B)$  and use (13) in order to compute  $\text{Var}_p(\mathcal{F}, C)$  for a multi-branched invariant curve  $C$ .

The variation index satisfies a property of global nature expressed in the following terms: if  $\mathcal{F}$  is a foliation on a complex surface  $M$  and  $C \subset M$  is a compact invariant curve, then

$$(15) \quad \sum_{p \in \text{Sing} \mathcal{F} \cap C} \text{Var}_p(\mathcal{F}, C) = c_1(N_{\mathcal{F}}) \cdot C.$$



**3.5. The polar excess index.** Let  $\eta$  be a formal meromorphic 1-form with trivial divisor of zeros, written in coordinates  $(x, y)$  as

$$\eta = Pdx + Qdy,$$

where  $P, Q$  are formal meromorphic functions. For  $(a : b) \in \mathbb{P}^1$ , the polar curve of  $\eta$  with respect to  $(a : b)$  is the formal curve  $\mathcal{P}_{(a,b)}^\eta$  associated to the equation  $aP + bQ = 0$ . Let  $B$  be an irreducible curve, not contained in the pole divisor  $(\eta)_\infty$ , having  $\gamma(t)$  as a Puiseux parametrization. We say that  $B$  is *invariant* by  $\eta$  if  $\gamma^*\eta \equiv 0$ . In this case, we define the *polar intersection number* of  $\eta$  and  $B$  at  $p$  (see [6, 11]) as the generic value of

$$(\mathcal{P}^\eta, B)_p = (\mathcal{P}_{(a,b)}^\eta, B)_p = \text{ord}_{t=0}((aP + bQ) \circ \gamma)$$

for  $(a : b) \in \mathbb{P}^1$ . This is an ingredient for the following definition:

**Definition 3.1.** Let  $\mathcal{F}$  be a germ of singular foliation at  $(\mathbb{C}^2, p)$ . Let  $B$  be a branch of separatrix and  $\hat{F}$  be a reduced balanced equation of separatrices adapted to  $B$ . The *polar excess index* [6, 11] of  $\mathcal{F}$  with respect to  $B$  is the integer

$$\Delta_p(\mathcal{F}, B) = (\mathcal{P}^\mathcal{F}, B)_p - (\mathcal{P}^{d\hat{F}}, B)_p.$$

For a curve of separatrices  $C$ , with irreducible factors as  $B_1, \dots, B_r$ , we define the polar excess index in an additive way:

$$\Delta_p(\mathcal{F}, C) = \sum_{i=1}^r \Delta_p(\mathcal{F}, B_i) = \sum_{i=1}^r \left( (\mathcal{P}^\mathcal{F}, B_i)_p - (\mathcal{P}^{d\hat{F}}, B_i)_p \right).$$

This definition is independent of the balanced equation, so, in order to compute the polar excess for a multi-branched curve, a balanced equation simultaneously adapted to all its branches can be employed. The additive character of the  $\Delta$ -index enables us to extend its definition to an arbitrary divisor  $\mathcal{B} = \sum_B a_B \cdot B$  in  $\text{Div}_p(\mathcal{F})$ :

$$\Delta_p(\mathcal{F}, \mathcal{B}) = \sum_B a_B \Delta_p(\mathcal{F}, B).$$

We can also formulate the  $\Delta$ -index as the residue of the logarithmic derivative of the ratio of equations of polar curves for  $\mathcal{F}$  and for  $d\hat{F}$ , where  $\hat{F}$  is an irreducible balanced equation of separatrices adapted to the invariant curve. More precisely, if  $\omega = Pdx + Qdy$  induces  $\mathcal{F}$ , we define, for  $(a : b) \in \mathbb{P}^1$ , the formal meromorphic 1-form

$$\eta_{(a:b)} = \frac{a\hat{F}_x + b\hat{F}_y}{aP + bQ} d \left( \frac{aP + bQ}{a\hat{F}_x + b\hat{F}_y} \right) = \frac{d(aP + bQ)}{aP + bQ} - \frac{d(a\hat{F}_x + b\hat{F}_y)}{a\hat{F}_x + b\hat{F}_y}.$$

Then, for generic  $(a : b)$ ,

$$\Delta_p(\mathcal{F}, B) = \text{Res}_{t=0} \gamma^* \eta_{(a:b)}.$$

Moreover, if  $C$  is an  $\mathcal{F}$ -invariant analytic curve, then, still for generic  $(a : b)$ ,

$$\Delta_p(\mathcal{F}, C) = \frac{1}{2\pi i} \int_{\partial C} \eta_{(a:b)}.$$

The following simple calculations are done in [11] for an  $\mathcal{F}$ -invariant branch  $B$ :

- If  $\mathcal{F}$  is a non-singular then  $\Delta_p(\mathcal{F}, B) = 0$ .

- If  $\mathcal{F}$  has a non-degenerated reduced singularity, then

$$\Delta_p(\mathcal{F}, B) = 0.$$

- If  $\mathcal{F}$  has a saddle-node singularity with weak index  $k + 1$  we have two possibilities: either  $\Delta_p(\mathcal{F}, B) = 0$ , when  $B$  is the strong separatrix, or  $\Delta_p(\mathcal{F}, B) = k > 0$ , when  $B$  is the weak separatrix.

In general, taking into account the behavior of the  $\Delta$ -index under blow-ups (equation (18) below), we have

$$(16) \quad \Delta_p(\mathcal{F}, B) = \begin{cases} \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(B) \tau_q(\mathcal{F}) & \text{if } B \text{ is a strong or dicritical separatrix} \\ k + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \nu_q(B) \tau_q(\mathcal{F}) & \text{if } B \text{ is a weak separatrix,} \end{cases}$$

where  $k + 1$  is the weak index associated to  $B$ . Thus,  $\Delta_p(\mathcal{F}, B) \geq 0$  and  $\Delta_p(\mathcal{F}, B) = 0$  if and only if  $\mathcal{F}$  is a second class foliation and  $B$  is a strong or dicritical separatrix. The polar excess index is a measure of the existence of saddle-nodes singularities in the desingularization of  $\mathcal{F}$ . This interpretation derives from the following result of [11], which is a consequence of formula (16):

**Theorem 3.2.** *If  $\mathcal{F}$  is a germ of singular foliation at  $(\mathbb{C}^2, p)$  and  $C$  is a curve of separatrices, then  $\Delta_p(\mathcal{F}, C) \geq 0$ . Moreover, if  $\mathcal{B}$  is a balanced divisor of separatrices of  $\mathcal{F}$ , then  $\mathcal{F}$  is generalized curve foliation if and only if*

$$\Delta_p(\mathcal{F}, \mathcal{B}_0) = 0.$$

The polar excess and the GSV-index are interrelated by the following result [11]:

**Theorem 3.3.** *Let  $\mathcal{F}$  be a germ of singular foliation at  $(\mathbb{C}^2, p)$ . Let  $C$  be a curve of separatrices and  $\mathcal{B}$  be a balanced divisor adapted to  $C$ . Then*

$$\text{GSV}_p(\mathcal{F}, C) = \Delta_p(\mathcal{F}, C) + (C, \mathcal{B}_0 - C)_p - (C, \mathcal{B}_\infty)_p.$$

*In particular, when  $\mathcal{F}$  is non-dicritical and  $C$  is the complete set of separatrices, then*

$$\text{GSV}_p(\mathcal{F}, C) = \Delta_p(\mathcal{F}, C).$$

**3.6. The total index.** Let  $I_p(\mathcal{F}, C)$  denote one of the four residue-type indices relative to a curve of separatrices  $C$  defined so far — CS, GSV, Var or  $\Delta$ . When  $\mathcal{F}$  is non-dicritical and  $C$  is the complete set of separatrices, it is usual to say that  $I_p(\mathcal{F}, C)$  is *total*. When it comes to dicritical singularities, an attempt to establish a definition of total index involves the choice of a finite subset of  $\text{Sep}_p(\mathcal{F})$  as a reference. We propose to use balanced divisors of separatrices for this goal:

**Definition 3.4.** Let  $\mathcal{F}$  be a foliation at  $(\mathbb{C}^2, p)$  and  $\mathcal{B}$  be a primitive balanced divisor of separatrices. The *total* index of  $\mathcal{F}$  at  $p$  is defined as

$$I_p(\mathcal{F}, \mathcal{B}_0) - I_p(\mathcal{F}, \mathcal{B}_\infty)$$

and denoted by

$$I_p(\mathcal{F}) = \text{CS}_p(\mathcal{F}), \text{GSV}_p(\mathcal{F}), \text{Var}_p(\mathcal{F}) \text{ or } \Delta_p(\mathcal{F}).$$

Observe that  $I_p(\mathcal{F}, B)$  is the same for all branches  $B \in \text{Sep}_p(D)$  associated to the same dicritical component  $D \subset \mathcal{D}$ . This results from formula (16) for the  $\Delta$ -index and, for the three other indices, from similar formulas based on their behavior under blow-ups (see [3]). As a consequence,  $I_p(\mathcal{F})$  does not depend on the choice of the primitive balanced divisor. We inherit a connecting relation similar to (14):

$$\text{Var}_p(\mathcal{F}) = \text{CS}_p(\mathcal{F}) + \text{GSV}_p(\mathcal{F}).$$

The total Var and the total  $\Delta$  indices may be calculated using any balanced divisor of separatrices  $\mathcal{B}$ , not necessarily a reduced one:

$$\text{Var}_p(\mathcal{F}) = \text{Var}_p(\mathcal{F}, \mathcal{B}) \quad \text{and} \quad \Delta_p(\mathcal{F}) = \Delta_p(\mathcal{F}, \mathcal{B}).$$

Next we state a slightly modified version of Theorem 3.2 involving the total  $\Delta$ . We remark that, when the desingularization divisor  $\mathcal{D}$  of  $\mathcal{F}$  is devoid of dicritical components of valence two or higher, there are no poles in a primitive balance divisor of separatrices and the statement below is precisely that of Theorem 3.2.

**Proposition 3.5.**  *$\mathcal{F}$  is a generalized curve foliation at  $(\mathbb{C}^2, p)$  if and only if  $\Delta_p(\mathcal{F}) = 0$ .*

*Proof.* A generalized curve foliation is in particular second class and all its separatrices are either strong or dicritical. Thus, formula (16) gives  $\Delta_p(\mathcal{F}, B) = 0$  for every  $B \in \text{Sep}_p(\mathcal{F})$ , which on its turn implies that  $\Delta_p(\mathcal{F}, \mathcal{B}) = 0$  for any divisor of separatrices  $\mathcal{B}$  and, in particular, for a balanced divisor.

The converse proof is based on the following fact: if  $\mathcal{D}$  is the desingularization divisor of  $\mathcal{F}$ , then there is at least one isolated separatrix crossing each component of the  $\tilde{\mathcal{F}}$ -invariant part of  $\mathcal{D}$  [18, Prop. 4]. Thus, the number of isolated separatrices is at least  $1 + \sum_D (\text{Val}(D) - 1)$ , where the sum is over all dicritical components  $D \subset \mathcal{D}$ . Let  $\mathcal{B}$  be a primitive balanced divisor and  $D \subset \mathcal{D}$  be a dicritical component of  $\text{Val}(D) > 2$ . The pole divisor  $\mathcal{B}_\infty$  contains  $\text{Val}(D) - 2$  separatrices of  $\text{Sep}_p(D)$ . Note that  $D$  appears in the desingularization process as a component of valence 0, 1, or 2 (when it results, respectively, from the blow-up at  $p$  itself, at a non corner singularity or at a corner singularity). So at least  $\text{Val}(D) - 2$  points of  $D$  will be blown-up in the subsequent steps of the reduction process and to each one of them we can associate an isolated separatrix. Therefore, to each dicritical separatrix  $B$  appearing in  $\mathcal{B}_\infty$ , we can associate in an injective way one such isolated separatrix  $\tilde{B}$ . It follows from (16) that

$$(17) \quad \Delta_p(\mathcal{F}, B) \leq \Delta_p(\mathcal{F}, \tilde{B}).$$

Denote by  $\tilde{\mathcal{B}}_0$  the divisor obtained by summing up these  $\tilde{B}$ . We have  $\Delta_p(\mathcal{F}, \tilde{\mathcal{B}}_0 - \mathcal{B}_\infty) \geq 0$  by (17). Now we decompose  $\mathcal{B}_0 = \hat{\mathcal{B}}_0 + \tilde{\mathcal{B}}_0$  as a sum of effective divisor, where  $\hat{\mathcal{B}}_0$  is non-trivial. Then

$$0 = \Delta_p(\mathcal{F}, \mathcal{B}) = \Delta_p(\mathcal{F}, \mathcal{B}_0) - \Delta_p(\mathcal{F}, \mathcal{B}_\infty) = \Delta_p(\mathcal{F}, \hat{\mathcal{B}}_0) + \Delta_p(\mathcal{F}, \tilde{\mathcal{B}}_0 - \mathcal{B}_\infty).$$

The terms at the right are non negative and thus both are zero. This implies, in particular, that  $\Delta_p(\mathcal{F}, B) = 0$  for every separatrix  $B$  in  $\hat{\mathcal{B}}_0$ . Formula (16) then gives at once that  $\mathcal{F}$  is a second type foliation and that every isolated  $B$  in  $\hat{\mathcal{B}}_0$  is a weak separatrix. For the separatrices in  $\tilde{\mathcal{B}}_0$ , remark that each inequality (17) is actually an equality, and this is possible only if  $\tilde{B}$  is a strong separatrix. Summarizing,  $\mathcal{F}$  is a second class foliation having only strong isolated separatrices. It is therefore a generalized curve foliation.  $\square$

## 4. SECOND VARIATION INDEX

In order to condense notation and terminology, we assemble the variation and the polar excess indices in a new invariant:

**Definition 4.1.** Let  $\mathcal{F}$  be a germ of singular foliation at  $(\mathbb{C}^2, p)$  and  $C$  be a curve of separatrices. The *second variation index* of  $\mathcal{F}$  along  $C$  is defined as

$$\zeta_p(\mathcal{F}, C) = \text{Var}_p(\mathcal{F}, C) + \Delta_p(\mathcal{F}, C).$$

The variation and the polar excess are additive in the separatrices and this property is inherited by the  $\zeta$ -index. We can therefore define it for a divisor of separatrices and have a *total second variation index* by means of a balanced divisor  $\mathcal{B}$ :

$$\zeta_p(\mathcal{F}) = \zeta_p(\mathcal{F}, \mathcal{B}) = \text{Var}_p(\mathcal{F}) + \Delta_p(\mathcal{F}).$$

Next, we describe the behavior of the second variation under a blow-up  $\sigma : (\tilde{\mathbb{C}}^2, D) \rightarrow (\mathbb{C}^2, p)$ . As usual, we denote respectively by  $\sigma^*\mathcal{F} = \tilde{\mathcal{F}}$  and  $\sigma^*B = \tilde{B}$  the transforms of the foliation  $\mathcal{F}$  and of a branch of separatrix  $B \in \text{Sep}_p(\mathcal{F})$ . A divisor of separatrices  $\mathcal{B} = \sum_B a_B \cdot B$  is said to be of *order*  $q \in D$  if  $\tilde{B} \cap D = q$  whenever  $a_B \neq 0$ . If this is so, the transform of  $\mathcal{B}$  is defined as  $\tilde{\mathcal{B}} = \sum_B a_B \cdot \tilde{B}$ , which is a divisor of separatrices for  $\tilde{\mathcal{F}}$  at  $q \in D$ .

**Lemma 4.2.** *With the notation above, if  $q = \tilde{B} \cap D$ , then*

$$\zeta_q(\tilde{\mathcal{F}}, \tilde{B}) = \zeta_p(\mathcal{F}, B) - (m_p(\mathcal{F}) + \tau_p(\mathcal{F}))\nu_p(B),$$

where  $\tau_p(\mathcal{F})$  is the tangency excess of  $\mathcal{F}$  at  $p$  and

$$m_p(\mathcal{F}) = \begin{cases} \nu_p(\mathcal{F}), & \text{if } \sigma \text{ is non-dicritical} \\ \nu_p(\mathcal{F}) + 1, & \text{if } \sigma \text{ is dicritical.} \end{cases}$$

Moreover, if  $\mathcal{B}$  is a divisor of separatrices of order  $q \in D$ , then

$$\zeta_q(\tilde{\mathcal{F}}, \tilde{\mathcal{B}}) = \zeta_p(\mathcal{F}, \mathcal{B}) - (m_p(\mathcal{F}) + \tau_p(\mathcal{F}))\nu_p(\mathcal{B}).$$

*Proof.* The formula for a branch  $B \subset \text{Sep}_p(\mathcal{F})$  is a consequence of known formulas for the behavior under blow-ups for the variation [3] and the polar excess [11] indices:

$$(18) \quad \begin{aligned} \text{Var}_q(\tilde{\mathcal{F}}, \tilde{B}) &= \text{Var}_p(\mathcal{F}, B) - m_p(\mathcal{F})\nu_p(B), \\ \Delta_q(\tilde{\mathcal{F}}, \tilde{B}) &= \Delta_p(\mathcal{F}, B) - \tau_p(\mathcal{F})\nu_p(B). \end{aligned}$$

The expression for a divisor is then a consequence of the additiveness of the second variation index.  $\square$

Now we examine the total second variation. We have that  $\zeta_p(\mathcal{F}) = \zeta_p(\mathcal{F}, \mathcal{B})$ , where  $\mathcal{B}$  is a balanced divisor of separatrices. Suppose that the  $D$ -singular point of  $\tilde{\mathcal{F}}$  are  $q_1, \dots, q_\ell$ . In order to calculate the total  $\zeta$  at these points we need to relate the transform of  $\mathcal{B}$  with balanced divisors at the points  $q_j$ . Denote by  $S(q_j) \subset \text{Sep}_p(\mathcal{F})$  the subset of all separatrices of order  $q_j \in D$  and decompose

$$\mathcal{B} = \sum_{B \in \text{Sep}_p(\mathcal{F})} a_B \cdot B = \sum_{j=1}^{\ell} \sum_{B \in S(q_j)} a_B \cdot B = \sum_{j=1}^{\ell} \mathcal{B}_j,$$

where  $\mathcal{B}_j = \sum_{B \in S(q_j)} a_B \cdot B$ . As before, denote by  $\tilde{\mathcal{B}}_j$  the transform of  $\mathcal{B}_j$ . There are two situations [11]:

- $\sigma$  is a non-dicritical blow-up, meaning that the exceptional divisor is  $\tilde{\mathcal{F}}$ -invariant. Then  $\tilde{\mathcal{B}}_j + D$  is a balanced divisor for  $\tilde{\mathcal{F}}$  at  $q_j$ , where we keep denoting by  $D$  the germ of the exceptional divisor at  $q_j$ .
- $\sigma$  is a dicritical blow-up, one such that the exceptional divisor is not  $\tilde{\mathcal{F}}$ -invariant. Then  $\tilde{\mathcal{B}}_j$  is a balanced divisor for  $\tilde{\mathcal{F}}$  at  $q_j$ .

We can state the following result:

**Proposition 4.3.** *Let  $\sigma : (\tilde{\mathbb{C}}^2, D) \rightarrow (\mathbb{C}^2, p)$  be a blow-up at  $p \in \mathbb{C}^2$ . Suppose that  $q_1, \dots, q_\ell$  are the  $D$ -singular points of  $\tilde{\mathcal{F}}$ . Then*

$$\zeta_p(\mathcal{F}) = \begin{cases} \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) + \nu_p^2(\mathcal{F}) - \tau_p^2(\mathcal{F}) & \text{if } \sigma \text{ is non-dicritical} \\ \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) + (\nu_p(\mathcal{F}) + 1)^2 - \tau_p^2(\mathcal{F}) & \text{if } \sigma \text{ is dicritical.} \end{cases}$$

*Proof.* We split the proof in two parts.

Part 1: *The non-dicritical case.* The total  $\zeta$  at each  $q_j$  is

$$(19) \quad \zeta_{q_j}(\tilde{\mathcal{F}}) = \zeta_{q_j}(\tilde{\mathcal{F}}, \tilde{\mathcal{B}}_j + D) = \zeta_{q_j}(\tilde{\mathcal{F}}, \tilde{\mathcal{B}}_j) + \zeta_{q_j}(\tilde{\mathcal{F}}, D).$$

We first calculate

$$(20) \quad \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}, D) = \sum_{j=1}^{\ell} \text{Var}_{q_j}(\tilde{\mathcal{F}}, D) + \sum_{j=1}^{\ell} \Delta_{q_j}(\tilde{\mathcal{F}}, D).$$

The sum of Var-indices along  $D$  is given by equation (15):

$$(21) \quad \sum_{j=1}^{\ell} \text{Var}_{q_j}(\tilde{\mathcal{F}}, D) = (c_1(N_{\tilde{\mathcal{F}}})) \cdot D = (-\nu_p(\mathcal{F}) D) \cdot D = \nu_p(\mathcal{F}).$$

On the other hand,  $\tilde{\mathcal{B}}_j + D$  is a balanced divisor of separatrices at  $q_j \in D$ . Thus, we get from Theorem 3.3 that

$$(22) \quad \sum_{j=1}^{\ell} \Delta_{q_j}(\tilde{\mathcal{F}}, D) = \sum_{j=1}^{\ell} \text{GSV}_{q_j}(\tilde{\mathcal{F}}, D) - \sum_{j=1}^{\ell} (D, \tilde{\mathcal{B}}_j)_{q_j}.$$

Now, we use (12) to compute the sum of GSV-indices along  $D$ :

$$\begin{aligned} \sum_{j=1}^{\ell} \text{GSV}_{q_j}(\tilde{\mathcal{F}}, D) &= c_1(N_{\tilde{\mathcal{F}}}) \cdot D - D \cdot D \\ &= (-\nu_p(\mathcal{F}) D) \cdot D - D \cdot D \\ &= \nu_p(\mathcal{F}) + 1. \end{aligned}$$

Since

$$\sum_{j=1}^{\ell} (D, \tilde{\mathcal{B}}_j)_{q_j} = \sum_{j=1}^{\ell} \nu_p(\mathcal{B}_j) = \nu_p(\mathcal{B}),$$

using Proposition 2.4, equation (22) turns into

$$(23) \quad \sum_{j=1}^{\ell} \Delta_{q_j}(\tilde{\mathcal{F}}, D) = \nu_p(\mathcal{F}) + 1 - \nu_p(\mathcal{B}) = \tau_p(\mathcal{F}).$$

It follows from (20), (21) and (23) that

$$(24) \quad \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}, D) = \nu_p(\mathcal{F}) + \tau_p(\mathcal{F}).$$

Combining (19) and (24), we find

$$(25) \quad \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) = \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}, \tilde{\mathcal{B}}_j) + \nu_p(\mathcal{F}) + \tau_p(\mathcal{F}).$$

Now Lemma 4.2 implies that

$$(26) \quad \begin{aligned} \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}, \tilde{\mathcal{B}}_j) &= \sum_{j=1}^{\ell} \zeta_p(\mathcal{F}, \mathcal{B}_j) - (\nu_p(\mathcal{F}) + \tau_p(\mathcal{F})) \sum_{j=1}^{\ell} \nu_p(\mathcal{B}_j) \\ &= \zeta_p(\mathcal{F}) - (\nu_p(\mathcal{F}) + \tau_p(\mathcal{F})) \nu_p(\mathcal{B}). \end{aligned}$$

From (25) and (26),

$$\begin{aligned} \zeta_p(\mathcal{F}) &= \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) + (\nu_p(\mathcal{F}) + \tau_p(\mathcal{F}))(\nu_p(\mathcal{B}) - 1) \\ &= \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) + (\nu_p(\mathcal{F}) + \tau_p(\mathcal{F}))(\nu_p(\mathcal{F}) - \tau_p(\mathcal{F})) \\ &= \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) + \nu_p^2(\mathcal{F}) - \tau_p^2(\mathcal{F}) \end{aligned}$$

and we are done.

Part 2: The dicritical case. Now  $\tilde{\mathcal{B}}_j$  is a balanced divisor of separatrices for  $\tilde{\mathcal{F}}$  at  $q_j$ . Then, it follows from Lemma 4.2 and Proposition 2.4 that

$$\begin{aligned}
\zeta_p(\mathcal{F}) &= \sum_{j=1}^{\ell} \zeta_{q_j}(\mathcal{F}, \mathcal{B}_j) \\
&= \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}, \tilde{\mathcal{B}}_j) + (\nu_p(\mathcal{F}) + 1 + \tau_p(\mathcal{F})) \sum_{j=1}^{\ell} \nu_p(\mathcal{B}_j) \\
&= \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) + (\nu_p(\mathcal{F}) + 1 + \tau_p(\mathcal{F})) \nu_p(\mathcal{B}) \\
&= \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) + (\nu_p(\mathcal{F}) + 1 + \tau_p(\mathcal{F}))(\nu_p(\mathcal{F}) + 1 - \tau_p(\mathcal{F})) \\
&= \sum_{j=1}^{\ell} \zeta_{q_j}(\tilde{\mathcal{F}}) + (\nu_p(\mathcal{F}) + 1)^2 - \tau_p^2(\mathcal{F}).
\end{aligned}$$

This completes the proof of the proposition.  $\square$

## 5. PROOF OF THEOREM I

In this section we compare second variation and Baum-Bott indices and achieve a proof for Theorem I. We start with a look at reduced singularities:

**Lemma 5.1.** *Let  $\mathcal{F}$  be a reduced germ of foliation at  $(\mathbb{C}^2, p)$ . Then*

$$\zeta_p(\mathcal{F}) = \text{BB}_p(\mathcal{F}).$$

*Proof.* A reduced foliation is non-dicritical and so  $\Delta_p(\mathcal{F}) = \text{GSV}_p(\mathcal{F})$ , which implies  $\zeta_p(\mathcal{F}) = \text{Var}_p(\mathcal{F}) + \text{GSV}_p(\mathcal{F})$ . We only need to assemble information from [3] (see the two examples on p. 538).

On the one hand, when  $p$  is non-degenerate with local model given by (3), we have:

$$\text{Var}_p(\mathcal{F}) = \text{BB}_p(\mathcal{F}) = \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2} + 2.$$

This implies our result, since  $\text{GSV}_p(\mathcal{F}) = 0$ .

On the other hand, for a saddle-node singularity, with normal form as in (4), we have

$$\text{Var}_p(\mathcal{F}) = k + 2 + \lambda \quad \text{and} \quad \text{GSV}_p(\mathcal{F}) = k,$$

while

$$\text{BB}_p(\mathcal{F}) = 2k + 2 + \lambda.$$

$\square$

In the non-reduced case, we have:

**Theorem 5.2.** *Let  $\mathcal{F}$  be a germ of singular foliation at  $(\mathbb{C}^2, p)$ . Then*

$$\text{BB}_p(\mathcal{F}) = \zeta_p(\mathcal{F}) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \tau_q(\mathcal{F})^2,$$

where the summation runs over all infinitely near points of  $\mathcal{F}$  at  $p$ .



*Proof.* We recall the behavior of the Baum-Bott index under blow-ups [3, Prop. 1]: if  $\sigma : (\tilde{\mathbb{C}}^2, D) \rightarrow (\mathbb{C}^2, p)$  is a blow-up at  $p \in \mathbb{C}^2$  and  $q_1, \dots, q_\ell$  are the  $D$ -singular points of  $\tilde{\mathcal{F}}$ , then

$$\text{BB}_p(\mathcal{F}) = \sum_{j=1}^{\ell} \text{BB}_{q_j}(\tilde{\mathcal{F}}) + c_1^2(N_{\tilde{\mathcal{F}}}).$$

This translates into

$$(27) \quad \text{BB}_p(\mathcal{F}) = \begin{cases} \sum_{j=1}^{\ell} \text{BB}_{q_j}(\tilde{\mathcal{F}}) + \nu_p^2(\mathcal{F}), & \text{if } \sigma \text{ is non-dicritical} \\ \sum_{j=1}^{\ell} \text{BB}_{q_j}(\tilde{\mathcal{F}}) + (\nu_p(\mathcal{F}) + 1)^2, & \text{if } \sigma \text{ is dicritical.} \end{cases}$$

Define

$$\vartheta_p(\mathcal{F}) = \text{BB}_p(\mathcal{F}) - \zeta_p(\mathcal{F}) - \sum_{q \in \mathcal{I}_p(\mathcal{F})} \tau_q(\mathcal{F})^2.$$

We first observe that, if  $\mathcal{F}$  is reduced, then  $\mathcal{I}_p(\mathcal{F}) = \{p\}$  and  $\tau_p(\mathcal{F}) = 0$ , resulting in  $\vartheta_p(\mathcal{F}) = 0$  by the application of Lemma 5.1. In general, if  $\mathcal{F}$  is non-reduced, for a blow-up  $\sigma$  as above, we take into account the decomposition

$$\sum_{q \in \mathcal{I}_p(\mathcal{F})} \tau_q^2(\mathcal{F}) = \tau_p^2(\mathcal{F}) + \sum_{j=1}^{\ell} \left( \sum_{q \in \mathcal{I}_{q_j}(\tilde{\mathcal{F}})} \tau_q^2(\tilde{\mathcal{F}}) \right)$$

along with Propositions 4.3 and formula (27) in order to conclude that

$$\vartheta_p(\mathcal{F}) = \sum_{j=1}^{\ell} \vartheta_{q_j}(\tilde{\mathcal{F}}).$$

Finally, an induction argument gives that  $\vartheta_p(\mathcal{F}) = 0$ , proving the theorem.  $\square$

We recall that  $\zeta_p(\mathcal{F}) = \text{Var}_p(\mathcal{F}) + \Delta_p(\mathcal{F})$  and  $\text{Var}_p(\mathcal{F}) = \text{CS}_p(\mathcal{F}) + \text{GSV}_p(\mathcal{F})$ . When  $\mathcal{F}$  is non-dicritical,  $\Delta_p(\mathcal{F}) = \text{GSV}_p(\mathcal{F})$  and the theorem reads:

**Corollary 5.3.** *If  $\mathcal{F}$  is non-dicritical, then*

$$\text{BB}_p(\mathcal{F}) = \text{CS}_p(\mathcal{F}) + 2\text{GSV}_p(\mathcal{F}) + \sum_{q \in \mathcal{I}_p(\mathcal{F})} \tau_q(\mathcal{F})^2.$$

Since both  $\Delta_p(\mathcal{F})$  and  $\text{GSV}_p(\mathcal{F})$  are integers, and  $\text{GSV}_p(\mathcal{F}) \geq 0$  when  $\mathcal{F}$  is non-dicritical, the following corollary turns evident from Theorem 5.2:

**Corollary 5.4.** *Let  $\mathcal{F}$  be a germ of foliation at  $p \in \mathbb{C}^2$ . Then*

$$\text{BB}_p(\mathcal{F}) - \text{CS}_p(\mathcal{F}) \in \mathbb{Z}.$$

*This integer is non-negative when  $\mathcal{F}$  is non-dicritical.*

Indeed, this corollary could also be proved by following Baum-Bott indices along the reduction of  $\mathcal{F}$  — equation (27) — and comparing them with CS-indices for the reduced singularities. Baum-Bott's Theorem (equation (6)) brings the following consequence for global foliations:

**Corollary 5.5.** *Let  $\mathcal{F}$  be a foliation on a compact surface  $M$ . Then*

$$\sum_{p \in \text{Sing}(\mathcal{F})} \text{CS}_p(\mathcal{F}) \in \mathbb{Z}.$$

We have now all elements to complete the proof of Theorem I:

*Proof.* (of Theorem I) The first statement follows straight from Theorem 5.2. If  $\mathcal{F}$  is of second type at  $p$ , so is it at all infinitely near points, implying  $\tau_q(\mathcal{F}) = 0$  for all  $q \in \mathcal{I}_p(\mathcal{F})$  and  $\text{BB}_p(\mathcal{F}) = \zeta_p(\mathcal{F}) = \text{Var}_p(\mathcal{F}) + \Delta_p(\mathcal{F})$ . Conversely, the equality of indices implies that the summation in Theorem 5.2 vanishes, giving, in particular, that  $\tau_p(\mathcal{F}) = 0$  and that  $\mathcal{F}$  is of second type. The second statement is then a consequence of Proposition 3.5.  $\square$

## 6. LOGARITHMIC FOLIATIONS ON THE COMPLEX PROJECTIVE PLANE

Let  $\mathcal{F}$  be a holomorphic foliation on the complex projective plane  $\mathbb{P}^2 = \mathbb{P}_{\mathbb{C}}^2$ . The degree of  $\mathcal{F}$  is the number  $\deg(\mathcal{F})$  of tangencies between  $\mathcal{F}$  and a generic line. The question concerning the existence of a bound for the degree  $\deg(S)$  of an  $\mathcal{F}$ -invariant curve  $S$  in terms of  $\deg(\mathcal{F})$  is known in foliation theory as Poincaré problem [20]. When all singularities of  $\mathcal{F}$  over  $S$  are non-dicritical, it is proven in [7] that the inequality  $\deg(S) \leq \deg(\mathcal{F}) + 2$  holds. The limit case for this bound is reached by *logarithmic foliations*, those defined by logarithmic 1-forms, as explained next. Suppose that an  $\mathcal{F}$ -invariant algebraic curve  $S \subset \mathbb{P}^2$  is defined by a homogeneous polynomial equation  $P = P_1 P_2 \cdots P_n = 0$ , where each polynomial  $P_i$  is irreducible of degree  $d_i$ . Suppose further that  $\mathcal{F}$  is non-dicritical at each point  $q \in S$ . Then the following statements are equivalent [3, 6, 8]:

- (1)  $\deg(S) = \deg(\mathcal{F}) + 2$ .
- (2) There are residues  $\lambda_i \in \mathbb{C}^*$  with  $\sum_{i=1}^n \lambda_i d_i = 0$  such that  $\mathcal{F}$  is given by  $\omega = 0$ , where  $\omega$  is the global closed logarithmic 1-form in  $\mathbb{P}_{\mathbb{C}}^2$  defined by

$$\omega = \sum_{i=1}^n \lambda_i \frac{dP_i}{P_i}.$$

- (3) The foliation  $\mathcal{F}$  is a generalized curve foliation at each  $p \in \text{Sing}(\mathcal{F}) \cap S$  and  $S$  contains all branches of  $\text{Sep}_p(\mathcal{F})$  at each  $p$ .

As an application of Theorem I, we propose the following characterization of non-dicritical logarithmic foliations:

**Proposition 6.1.** *Let  $\mathcal{F}$  be a holomorphic foliation on  $\mathbb{P}^2$ . Suppose that  $\mathcal{F}$  leaves invariant an algebraic curve  $S$  such that:*

- $\text{Sing}(\mathcal{F}) \subset S$ ;
- all points  $p \in \text{Sing}(\mathcal{F})$  are non-dicritical and of second type;
- $S$  contains all the local branches of  $\text{Sep}_p(\mathcal{F})$  at each  $p \in \text{Sing}(\mathcal{F})$ .

*Then  $\deg(S) = \deg(\mathcal{F}) + 2$  and  $\mathcal{F}$  is a logarithmic foliation.*

*Proof.* Denote by  $d_0 = \deg(S)$  and  $d = \deg(\mathcal{F})$ . On the one hand, by Baum-Bott's Theorem (equation (6)), we have

$$\sum_{p \in \text{Sing}(\mathcal{F})} \text{BB}_p(\mathcal{F}) = (d + 2)^2.$$

On the other hand, by Theorem I and formulas (10) and (12),

$$\begin{aligned} \sum_{p \in \text{Sing}(\mathcal{F})} \text{BB}_p(\mathcal{F}) &= \sum_{p \in \text{Sing}(\mathcal{F})} \text{CS}_p(\mathcal{F}) + 2\text{GSV}_p(\mathcal{F}) \\ &= d_0^2 + 2((d+2)d_0 - d_0^2) \\ &= 2(d+2)d_0 - d_0^2. \end{aligned}$$

These two equations give  $d_0 = d + 2$ , which implies that  $\mathcal{F}$  is logarithmic.  $\square$

Actually, Proposition 6.1 can be stated in a more general setting, in the spirit of [3] and [14], switching  $\mathbb{P}^2$  to a compact projective surface  $M$  with Picard group  $\text{Pic}(M) = \mathbb{Z}$ . We need a definition: a meromorphic 1-form  $\omega$  on a complex manifold  $M$  is *logarithmic* if both  $\omega$  and  $d\omega$  have simple poles over  $(\omega)_\infty$ . We can then state:

**Proposition 6.2.** *Let  $M$  be a compact projective surface with Picard group  $\text{Pic}(M) = \mathbb{Z}$ . Let  $\mathcal{F}$  be a holomorphic foliation on  $M$  that leaves invariant a compact curve  $S$  satisfying the conditions listed in Proposition 6.1. Then  $\mathcal{F}$  is induced by a closed logarithmic 1-form having simple poles over  $S$ .*

*Proof.* Summing up  $\text{BB}_p(\mathcal{F}) = \text{CS}_p(\mathcal{F}) + 2\text{GSV}_p(\mathcal{F})$  over all  $p \in \text{Sing}(\mathcal{F})$ , we find

$$c_1(N_{\mathcal{F}})^2 = S \cdot S + 2(c_1(N_{\mathcal{F}}) \cdot S - S \cdot S) \Rightarrow (c_1(N_{\mathcal{F}}) - c_1(\mathcal{O}_S))^2 = 0.$$

Since  $\text{Pic}(M) = \mathbb{Z}$ , the line bundle  $L = N_{\mathcal{F}}^* \otimes \mathcal{O}_S$  is trivial, that is,  $N_{\mathcal{F}} = \mathcal{O}_S$ . Now, the proof follows the steps of Proposition 10 in [3]. We have that  $\mathcal{F}$  is induced by a meromorphic 1-form  $\omega$  on  $M$  with empty zero divisor and whose pole divisor is  $S$  with order one. The comment preceding that result also works here: if  $\sigma$  is a blow-up at  $p \in \text{Sing}(\mathcal{F})$ , then  $\sigma^*\omega$  has a pole of first order over  $\sigma^{-1}(S)$ . This is because  $\mathcal{F}$  is non-dicritical and second type, implying that  $C$ , the complete curve of separatrices at  $p$ , satisfies  $\nu_0(\mathcal{F}) = \nu_0(C) - 1$  by Proposition 2.4. Finally, taking  $\pi : \tilde{M} \rightarrow M$  a desingularization for  $S$ , the curve  $\tilde{S} = \pi^{-1}(S)$  has normal crossings and  $\tilde{\omega} = \pi^*\omega$  has a simple pole over  $\tilde{S}$ . Since  $\tilde{S}$  is invariant by  $\tilde{\omega}$ , the exterior derivative  $d\tilde{\omega}$  also has a simple pole over  $\tilde{S}$ . That is,  $\tilde{\omega}$  is a logarithmic form and Deligne's Theorem [9] asserts that it is closed, giving that  $\omega$  is also closed.  $\square$

## 7. EXAMPLES

We present two examples that give a numerical illustration of our results.

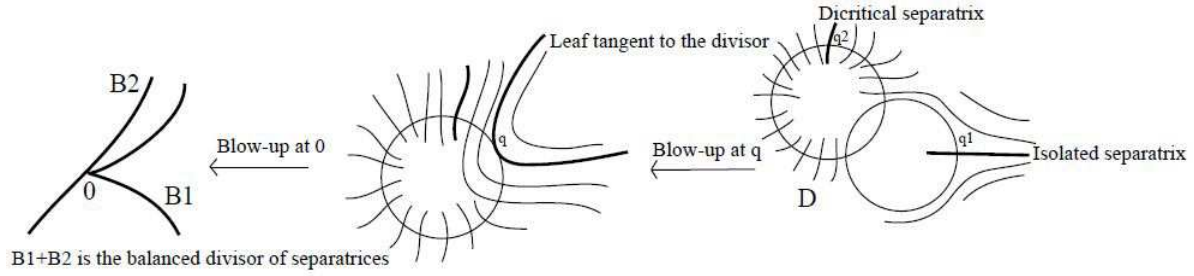
**Example 7.1** (Suzuki's example). Let  $\mathcal{F}$  be the germ of foliation at  $(\mathbb{C}^2, 0)$  defined by

$$\omega = (y^3 + y^2 - xy)dx - (2xy^2 + xy - x^2)dy.$$

$\mathcal{F}$  is a dicritical generalized curve foliation having the transcendental first integral

$$\frac{x}{y} \exp\left(\frac{y(y+1)}{x}\right)$$

and admitting no meromorphic first integral [23]. After one blow-up, the foliation is regular and has a unique leaf that is tangent to the exceptional divisor with tangency order one. It corresponds to the unique isolated separatrix  $B_1$ . The transverse leaves give rise to dicritical separatrices. Chose one of them and denote by  $B_2$  the corresponding dicritical



separatrix. Then  $\mathcal{B} = B_1 + B_2$  is a balanced divisor of separatrices. It follows from (27) that

$$\text{BB}_0(\mathcal{F}) = (\nu_0(\mathcal{F}) + 1)^2 = (2 + 1)^2 = 9.$$

The following simple calculation follow from (18):

$$\left. \begin{array}{l} \text{Var}_0(\mathcal{F}, B_1) = 6 \\ \text{Var}_0(\mathcal{F}, B_2) = 3 \end{array} \right\} \Rightarrow \text{Var}_0(\mathcal{F}) = \text{Var}_0(\mathcal{F}, B_1 + B_2) = 9.$$

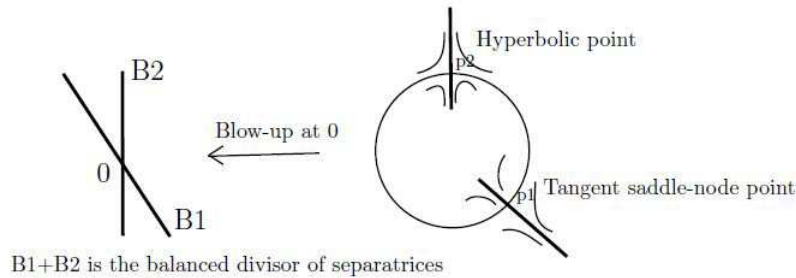
Since we are in the generalized curve case,  $\Delta_0(\mathcal{F}) = 0$  and Theorem I is verified.

**Example 7.2.** Let  $\mathcal{F}$  be the Ricatti foliation at  $(\mathbb{C}^2, 0)$  given by

$$\omega = (y^2 + xy + x^2)dx + x^2dy.$$

$\mathcal{F}$  is non-dicritical and has two separatrices  $B_1 : \{y = -x\}$  and  $B_2 : \{x = 0\}$ . After one blow-up, the foliation has two reduced singularities. The one corresponding to  $B_1$ , say  $p_1$ , is a tangent saddle-node with weak index 2. The other singularity,  $p_2$ , is hyperbolic with eigenvalue ratio  $-1$ . Therefore  $\mathcal{F}$  is not second type and  $\tau_0(\mathcal{F}) = 1$ . The divisor  $\mathcal{B} = B_1 + B_2$  is a balanced one. Simple calculations using (18) lead to:

$$\left. \begin{array}{l} \text{Var}_0(\mathcal{F}, B_1) = 3 \\ \text{Var}_0(\mathcal{F}, B_2) = 2 \end{array} \right\} \Rightarrow \text{Var}_0(\mathcal{F}) = \text{Var}_0(\mathcal{F}, B_1 + B_2) = 5$$



$$\left. \begin{array}{l} \Delta_0(\mathcal{F}, B_1) = 1 \\ \Delta_0(\mathcal{F}, B_2) = 1 \end{array} \right\} \Rightarrow \Delta_0(\mathcal{F}) = \Delta_0(\mathcal{F}, B_1 + B_2) = 2.$$

The expression of Theorem 5.2 is verified, since, from (27),

$$\text{BB}_0(\mathcal{F}) = \text{BB}_{p_1}(\mathcal{F}) + \text{BB}_{p_2}(\mathcal{F}) + \nu_0(\mathcal{F})^2 = 4 + 0 + 2^2 = 8.$$

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